# New Measures for the Quantization of Systems with Constraints*

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**Abstract**

Based on the results of a recent reexamination of the quantization of systems with first-class and second-class constraints from the point of view of coherent-state phase-space path integration, we give additional examples of the quantization procedure for reparameterization invariant Hamiltonians, for systems for which the original set of Lagrange multipliers are elevated to dynamical variables, as well as extend the formalism to include cases of first-class constraints the operator form of which have a spectral gap about the value zero that characterizes the quantum constraint subspace.

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# 1 Introduction

This paper presents several examples of quantization for dynamical systems possessing closed first-class constraints following general procedures established in Ref. [1]. For the reader's convenience we summarize several of the main ideas briefly; for a fuller discussion we suggest the reader consult [1].

A classical action (summation implied)

$$I = \textstyle\int [p_j \dot{q}^j - H(p,q) - \lambda^a \phi_a(p,q)]\, dt \;, \tag{1}$$

$1 \le j \le J$, $1 \le a \le A \le 2J$, is said to describe a system with closed first-class constraints provided $\{p_j, q^j\}$ are dynamical variables, $\{\lambda^a\}$ are Lagrange multipliers, and

$$\begin{aligned} \{\phi_a(p,q), \phi_b(p,q)\} &= c_{ab}{}^c\, \phi_c(p,q) \;, \\ \{\phi_a(p,q), H(p,q)\} &= h_a{}^b\, \phi_b(p,q) \;, \end{aligned} \tag{2}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket.

In the quantum theory we assume that ($\hbar = 1$)

$$\begin{aligned} [\Phi_a(P,Q), \Phi_b(P,Q)] &= i c_{ab}{}^c\, \Phi_c(P,Q) \;, \\ [\Phi_a(P,Q), \mathcal{H}(P,Q)] &= i h_a{}^b\, \Phi_b(P,Q) \;, \end{aligned} \tag{3}$$

where $\Phi_a$ and $\mathcal{H}$ denote self-adjoint constraint and Hamiltonian operators, respectively. In [1] a coherent-state path integral was defined (by a suitable lattice limit) in such a way that

$$\begin{aligned} \mathcal{M} \int \exp\{i \textstyle\int_0^T [p_j \dot{q}^j - H(p,q) - \lambda^a \phi_a(p,q)]\, dt\}\, \mathcal{D}p\, \mathcal{D}q\, \mathcal{D}C(\lambda) & \\ = \langle p'', q'' | e^{-i\mathcal{H}T}\, \mathbb{E} | p', q' \rangle [1 + O(\delta)] & \\ = \langle p'', q'' | \mathbb{E} e^{-i\mathcal{H}T}\, \mathbb{E} | p', q' \rangle [1 + O(\delta)] \;. & \end{aligned} \tag{4}$$

In this expression $\mathcal{M}$ is a formal normalization and $C(\lambda)$, $\int \mathcal{D}C(\lambda) = 1$, is a measure on the Lagrange multipliers designed to introduce (at least) one factor of a suitable *projection operator* $\mathbb{E}$. Because the set of self-adjoint operators $\{\Phi_a\}$ forms a closed Lie algebra, we may define $\exp(-i\xi^a\Phi_a)$, for appropriate sets $\{\xi^a\}$ of real parameters, as unitary group operators. For a *compact group* with a normalized Haar measure $\delta\xi$, $\int \delta\xi = 1$, then

$$\mathbb{E} = \textstyle\int e^{-i\xi^a \Phi_a}\, \delta\xi = \mathbb{E}(\Phi_a = 0) \;. \tag{5}$$

For a *noncompact group*, some generators of which have continuous spectra, then we choose a weight function $f(\xi)$ such that

$$\mathbb{E} = \textstyle\int e^{-i\xi^a \Phi_a}\, f(\xi)\, \delta\xi = \mathbb{E}(\Sigma\Phi_a^2 \leq \delta^2) \ , \tag{6}$$

for $0 < \delta \ll 1$. [Effectively, for a compact group, we may choose $\delta = 0$, and in that case the term $O(\delta) \equiv 0$ in both lines of (4).] As discussed in [1] the properly defined path integral with a suitable integration measure for the Lagrange multipliers—essentially different than in the Faddeev treatment [2]—leads, automatically, to gauge invariant results for compact groups, and after a suitable $\delta$-limiting process $\delta \to 0$ (see below) to gauge invariant results for noncompact groups. Moreover, no $\delta$-functionals of the classical constraints nor $\delta$-functionals of any subsidary conditions are introduced, and as a consequence no Faddeev-Popov determinant appears. Ambiguities that often may arise with such determinants [3] are thereby avoided at the outset.

In this paper we illustrate such quantization procedures for the example of a reparameterization invariant Hamiltonian which was not considered in [1] (see also [4]). We also consider the situation in which an original set of Lagrange multipliers are themselves elevated to the status of dynamical variables and used to define an extended dynamical system which is completed with the addition of suitable conjugates and new sets of constraints and their associated Lagrange multipliers. Finally, we extend the formalism to include rather general phase-space constraints the operator form of which has a spectral gap about zero. We generally follow the notation of [1].

# 2 Reparameterization Invariant Dynamics

Let us start with a single degree of freedom ($J = 1$) and the action

$$\textstyle\int [p\dot{q} - H(p, q)]\, dt \ . \tag{7}$$

We next promote the independent variable $t$ to a dynamical variable, introduce $s$ as its conjugate momentum (often called $p_t$), enforce the constraint $s + H(p, q) = 0$, and lastly introduce $\tau$ as a new independent variable. This modification is realized by means of the classical action

$$\textstyle\int \{pq' + st' - \lambda[s + H(p, q)]\}\, d\tau \ , \tag{8}$$

where $q' = dq/d\tau$, $t' = dt/d\tau$, and $\lambda = \lambda(\tau)$ is a Lagrange multiplier. The coherent-state path integral is constructed so that

$$\begin{aligned}\mathcal{M}\int \exp\bigl(i{\textstyle\int}\{pq' + st' - \lambda[s + H(p,q)]\}\,dt\bigr)\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}s\,\mathcal{D}t\,\mathcal{D}C(\lambda)& \\ = \langle p'', q'', s'', t''|\mathbb{E}|p', q', s', t'\rangle\ ,& \qquad (9)\end{aligned}$$

where

$$\begin{aligned}\mathbb{E} &\equiv \int_{-\infty}^{\infty} e^{-i\xi[S+\mathcal{H}(P,Q)]}\,\frac{\sin(\delta\xi)}{\pi\xi}\,d\xi \\ &= \mathbb{E}(-\delta < S + \mathcal{H}(P,Q) < \delta)\ . \qquad (10)\end{aligned}$$

The result in (9) and (10) represents as far as we can go without choosing $\mathcal{H}(P,Q)$.

To gain further insight into such expressions we specialize to the case of the free particle, $\mathcal{H} = P^2/2$. Then it follows [1] that

$$\begin{aligned}&\langle p'', q'', s'', t''|\mathbb{E}|p', q', s', t'\rangle \\ &= \pi^{-1}\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(k-p'')^2 - \tfrac{1}{2}(\tfrac{1}{2}k^2 + s'')^2 \\ &\qquad +ik(q''-q') - i\tfrac{1}{2}k^2(t''-t') \\ &\qquad -\tfrac{1}{2}(k-p')^2 - \tfrac{1}{2}(\tfrac{1}{2}k^2 + s')^2]\,dk \\ &\qquad \times\frac{2\sin[\delta(t''-t')]}{(t''-t')} + O(\delta^2)\ . \qquad (11)\end{aligned}$$

For any $\delta$ such that $0 < \delta \ll 1$, we observe that this expression represents a *reproducing kernel* which in turn defines an associated *reproducing kernel Hilbert space* composed of bounded, continuous functions given, for arbitrary complex numbers $\{\alpha_k\}$, phase-space points $\{p_k, q_k, s_k, t_k\}$, and $K < \infty$, by

$$\psi(p,q,s,t) = \sum_{k=0}^{K} \alpha_k \langle p, q, s, t|\mathbb{E}|p_k, q_k, s_k, t_k\rangle\ , \qquad (12)$$

or as the limit of Cauchy sequences of such functions in the norm defined by means of the inner product given by

$$(\psi, \psi) = {\textstyle\int} |\psi(p,q,s,t)|^2\,dp\,dq\,ds\,dt/(2\pi)^2 \qquad (13)$$

integrated over $\mathbb{R}^4$.

Let us next consider the *reduction* [1] of the reproducing kernel given by

$$\begin{aligned}
&\langle p'', q'', t''|p', q', t'\rangle \\
&\qquad \equiv \lim_{\delta\to 0} \frac{1}{4\sqrt{\pi}\delta} \int \langle p'', q'', s'', t''|\mathbb{E}|p', q', s', t'\rangle \, ds''\, ds' \\
&\qquad = \pi^{-1/2} \int \exp[-\tfrac{1}{2}(k-p'')^2 - \tfrac{1}{2}(k-p')^2 \\
&\qquad\qquad\qquad + ik(q''-q') - i\tfrac{1}{2}k^2(t''-t')]\, dk \;, \qquad (14)
\end{aligned}$$

which in turn generates a *new* reproducing kernel in the indicated variables. For the resultant kernel it is straightforward to demonstrate, for any $t$, that

$$\int \langle p'', q'', t''|p, q, t\rangle\langle p, q, t|p', q', t'\rangle \, dp\, dq/(2\pi) = \langle p'', q'', t''|p', q', t'\rangle \;. \qquad (15)$$

This relation implies that the span of the vectors $\{|p,q\rangle \equiv |p,q,0\rangle\}$ is identical with the span of the vectors $\{|p,q,t\rangle\}$, meaning further that the states $\{|p,q,t\rangle\}$ form a set of *extended coherent states*, which are extended with respect to $t$ in the sense of Ref. [5]. Observe how the time variable has become distinguished by the criterion (15). Consequently, we may properly interpret

$$\langle p'', q'', t''|p', q', t'\rangle \equiv \langle p'', q''|e^{-i(P^2/2)(t''-t')}|p', q'\rangle \;, \qquad (16)$$

namely as the conventional, single degree of freedom, coherent state matrix element of the evolution operator appropriate to the free particle.

To further demonstrate this interpretation as the dynamics of the free particle, we may pass to sharp $q$ matrix elements with the observation that

$$\begin{aligned}
&\langle q''|e^{-i(P^2/2)(t''-t')}|q'\rangle \\
&\qquad \equiv \frac{\pi^{1/2}}{(2\pi)^2} \int \langle p'', q''|e^{-i(P^2/2)(t''-t')}|p', q'\rangle \, dp''\, dp' \\
&\qquad = \frac{1}{2\pi} \int \exp[\, ik(q''-q') - i\tfrac{1}{2}k^2(t''-t')]\, dk \\
&\qquad = \frac{e^{i(q''-q')^2/2(t''-t')}}{\sqrt{2\pi i(t''-t')}} \;, \qquad (17)
\end{aligned}$$

which is evidently the usual result.

# 3 Elevating the Lagrange Multiplier to an Additional Dynamical Variable

Sometimes it is useful to consider an alternative formulation of a system with constraints in which the initial Lagrange multipliers are regarded as dynamical variables, complete with conjugate variables, and to introduce new constraints as needed. For example, let us start with a single degree of freedom system with a single first-class constraint specified by the action functional

$$\textstyle\int [p\dot{q} - H(p,q) - \lambda\phi(p,q)]\, dt \;, \tag{18}$$

where $\phi(p,q)$ represents the constraint and $\lambda$ the Lagrange multiplier. Instead, let us replace this action functional by

$$\textstyle\int [p\dot{q} + \pi\dot{\lambda} - H(p,q) - \xi\pi - \theta\phi(p,q)]\, dt \;. \tag{19}$$

In this expression we have introduced $\pi$ as the canonical conjugate to $\lambda$, the Lagrange multiplier $\xi$ to enforce the constraint $\pi = 0$, and the Lagrange multiplier $\theta$ to enforce the original constraint $\phi = 0$. Observe that $\{\pi, \phi(p,q)\} = 0$, and therefore the constraints remain first class in the new form. The path integral expression for the extended form reads

$$\begin{aligned} \mathcal{M} \int \exp\{i\textstyle\int [p\dot{q} + \pi\dot{\lambda} - H(p,q) - \xi\pi - \theta\phi(p,q)]\, dt\}\, \mathcal{D}p\, \mathcal{D}q\, \mathcal{D}\pi\, \mathcal{D}\lambda\, \mathcal{D}C(\xi,\theta) \\ = \langle p'', q'', \pi'', \lambda'' | e^{-i\mathcal{H}T} \mathbb{E} | p', q', \pi', \lambda' \rangle \;. \end{aligned} \tag{20}$$

In this expression

$$\mathbb{E} = \mathbb{E}(-\delta < \Phi(P,Q) < \delta)\, \mathbb{E}(-\delta < \Pi < \delta) \;. \tag{21}$$

Consequently, the complete propagator factors into two terms,

$$\begin{aligned} &\langle p'', q'', \pi'', \lambda'' | e^{-i\mathcal{H}T} \mathbb{E} | p', q', \pi', \lambda' \rangle \\ &\qquad = \langle p'', q'' | e^{-i\mathcal{H}T} \mathbb{E}(-\delta < \Phi(P,Q) < \delta) | p', q' \rangle \\ &\qquad \times \langle \pi'', \lambda'' | \mathbb{E}(-\delta < \Pi < \delta) | \pi', \lambda' \rangle \;. \end{aligned} \tag{22}$$

The first factor is exactly what would be found by the appropriate path integral of the original classical system with only the single constraint $\phi(p,q) = 0$

and the single Lagrange multiplier $\lambda$. The second factor represents the modification introduced by considering the extended system. Note however that with a suitable $\delta$-limit [1] the second factor reduces to a product of terms, one depending on the " $''$ " arguments, the other depending on the " $'$ " arguments. This result for the second factor implies that it has become the reproducing kernel for a *one-dimensional* Hilbert space, and when multiplied by the first factor it may be ignored entirely. In this way it is found that the quantization of the original and extended systems leads to identical results.

# 4 Constraints with a Spectral Gap about Zero

Let us initially consider the example of a classical constraint of the form

$$\phi(p,q) = p^2 + q^2 + q^4 - c = 0 \tag{23}$$

for a suitable value of $c$. Here the natural operator form of the constraint has a discrete spectrum, but a spectrum without any particular regularity (in contrast with the case where the quartic term is absent). The integral in (6) does not define the desired projection operator for any locally integrable $f$. Thus we must entertain the idea that the integral designed to yield a projection operator may in fact not always do so, and consequently, we need to develop a formulation in which the relevant integral yields the desired projection operator *plus an arbitrarily small correction term* that must be sent to zero in some subsequent limiting operation. In particular, motivated by the case at hand, let us consider the expression

$$\mathbb{F} \equiv \textstyle\int e^{-i\xi\Phi}\, f(\xi)\, d\xi \ , \tag{24}$$

where the spectrum of $\Phi$ is purely discrete, includes zero, and has a gap between the eigenvalue zero and the next closest eigenvalue. In particular, if $\Phi|m\rangle = \phi_{(m)}|m\rangle$ characterizes a complete set of eigenvectors and eigenvalues, then, excluding the particular value $\phi_{(m)} = 0$, we set $\Delta \equiv \inf(|\phi_{(m)}|)$ and assume that $\Delta > 0$. It is immaterial whether or not there are degeneracies. As noted, one of the eigenvalues $\phi_{(m)}$ is zero, and it is onto the subspace of that eigenvalue that we want to project. We let $\mathbb{E}$ denote the projection operator onto the desired subspace. Clearly, if we choose the integration in (24) as an average over the interval from $[-L, L]$, $L > 0$, then

$$\mathbb{F} = \mathbb{E} + W \tag{25}$$

where $W$ is a bounded operator with a bound given by

$$\|W\| \le \frac{1}{L\Delta} \; . \tag{26}$$

The operator $W$ is not a projection operator, but its operator norm can be made arbitrarily small by choosing $L$ large enough. In other words, $\mathbb{F} = \mathbb{E}$ apart from an operator of arbitrarily small norm.

For the case of closed first-class constraints, only a single projection operator is needed to effect the projection of the system onto the physical Hilbert space. In this case we may easily generalize the discussion of [1] to arrive at the conclusion that there exists a measure $L(\xi)$, $\textstyle\int dL(\xi) = 1$, on the Lagrange multipliers at the initial time such that, heuristically,

$$\begin{aligned} \mathcal{M} \int \exp\{i\textstyle\int[p_j\dot{q}^j - H(p,q) - \lambda^a\phi_a(p,q)]\,dt - i\xi^a\phi_a(p',q')\}\,\mathcal{D}p\,\mathcal{D}q\,dL(\xi) \\ = \langle p'', q''|e^{-i\mathcal{H}T}\mathbb{E}|p', q'\rangle[1 + O(L^{-1})] \; . \end{aligned} \tag{27}$$

Here the measure $L(\xi)$—and thereby the projection operator $\mathbb{E}$—are adequately defined by the requirement that

$$\mathbb{F} \equiv \textstyle\int e^{-i\xi^a\Phi_a}\,dL(\xi) \equiv \mathbb{E} + W \; , \tag{28}$$

where $\mathbb{E}$ represents the projection operator onto the appropriate quantum constraint subspace and $W$ is an operator such that $\|W\| \le (L\Delta)^{-1}$, for some $\Delta > 0$. Just as was the case for the $\delta$-limiting procedure, a subsequent limit as $L \to \infty$ can be introduced to the reproducing kernel, leading, in general, to a reduction of the appropriate reproducing kernel in the sense of [1]. On the other hand, observe that the original path integral (27) (with $L < \infty$) defines a reproducing kernel with an inner product having a local integral representation exactly as for the unconstrained system, and furthermore, if $L$ is huge, e.g., $L\Delta = 10^{50}$, then the errors represented by $W$ are negligible.

**Constraints with mixed spectra**

Although we have concentrated on constraint operators with purely discrete spectra, all that was really essential in showing that $\mathbb{F} - \mathbb{E}$ was an operator with a small norm was the existence of a gap in the spectrum at zero which we have termed $\Delta$ with $\Delta > 0$. So long as this gap remains it is possible to

argue that $\mathbb{F} - \mathbb{E}$ remains small even though the constraint operators may have continuous spectra away from zero. Thus without further comment we may extend the results of this section to include such cases. For example, in this extension one may consider the classical constraint given by $p^2 - 1/(1+q^2)^{3/2} + q^2/(1+q^2) = 0$ which in its quantum form has a normalizable eigenstate that satisfies the quantum constraint condition, but also has a continuous spectrum separated by an appropriate gap.


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